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Application of supersymmetric WKB method to cyclic shape invariant potentials

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Abstract

We examine the accuracy and the validity of the lowest order supersymmetric WKB (SWKB) formula for cyclic shape invariant potentials (CSIPs). For period-2 CSIPs, we show analytically that the SWKB formula can yield exact eigenenergies for either all the even states or all the odd states. Such alternate exactness of the SWKB formula is due to the fact that period-2 CSIPs can also be considered as a kind of translational shape invariant potential. For CSIPs with periods greater than 2, we note that the accuracy of the SWKB formula also demonstrates similar alternating patterns. However, as a consequence of the rapid oscillations in the potential at large distances, the SWKB quantization formula fails to produce highly accurate results even in the high-energy limit.

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1. Introduction

Supersymmetric quantum mechanics (SQM) (see, e.g., [1–6] and references therein), first proposed by Witten to illustrate dynamical breaking of supersymmetry (SUSY) in quantum field theory [1], has become an intriguing topic in its own right since the early 1980s. A host of related issues, such as shape invariant potentials (SIPs) [7–9], isospectral Hamiltonians [10–12], reflectionless potentials [13–15] and supersymmetric WKB (SWKB) approximation [16–20] have been discovered and discussed in the context SQM. The basic idea of SQM is to apply the factorization method [21–23] to express a given Hamiltonian $\hat{H}_1$ in an operator form (in units of $2m = \hbar = 1$),

$$\hat{H}_1 = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x)\right] = A^\dagger A. \tag{1.1}$$

The two operators A and $A^\dagger$ are related to a superpotential W,

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \tag{1.2}$$

$$A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x), \tag{1.3}$$

with W satisfying the equation

$$V_1(x) = W^2(x) - W'(x). \tag{1.4}$$

Hereafter, we use a prime to denote differentiation with respect to x. The merit of SQM is to relate the spectrum of $\hat{H}_1$ to that of another Hamiltonian $\hat{H}_2$ given by

$$\hat{H}_2 = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_2(x)\right] = AA^\dagger, \tag{1.5}$$

where

$$V_2(x) = W^2(x) + W'(x). \tag{1.6}$$

The eigenenergies of these two supersymmetric partner Hamiltonians, $E_0^{(i)} < E_1^{(i)} < E_2^{(i)} < \cdots$ $(i = 1, 2)$, are almost identical in the sense that

$$E_{n+1}^{(1)} = E_n^{(2)} \tag{1.7}$$

for unbroken SUSY where $E_0^{(1)} = 0$ and $A\psi_0^{(1)} = 0$, with $\psi_0^{(1)}$ being the ground-state wavefunction of $\hat{H}_1$.

For SIPs satisfying the shape invariant relation

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1), \tag{1.8}$$

where a_1, a_2 are free parameters characterizing the partner potentials V_1, V_2 and $a_2 = f(a_1)$, the entire spectrum of $\hat{H}_1$ is given by

$$E_0^{(1)} = 0, \qquad E_n^{(1)} = \sum_{k=1}^{n} R(a_k) \qquad \text{for} \quad n \geqslant 1, \tag{1.9}$$

with $a_k \equiv f^{k-1}(a_1)$. SIPs play an very important role in SQM because both the energies and wavefunctions of these system can be exactly obtained with algebraic methods. As a matter of fact, many exactly solvable potentials frequently encountered in mathematical physics belong to the class of SIPs (see, e.g., [4, 6, 24]). SIPs can be categorized according to the nature of the function $f(a_1)$ (see, e.g., [6]). If $f(a_1) = a_1 + \alpha$, where α is an arbitrary constant, the family of SIPs is said to be related by translation, termed as translational SIPs (TSIPs). On the other hand, if $f(a_1) = \alpha a_1$, such SIPs are related by scaling and called scaling SIPs (SSIPs) [8, 9].

Another kind of SIP discovered more recently is the cyclic shape invariant potential (CSIP) [25–27]. For CSIPs, the parameters a_1 and a_2 in (1.8) are related by a cyclic function f of period-p such that

$$a_{p+1} = f^p(a_1) = a_1, \tag{1.10}$$

with p being a positive integer. For period-2 CSIP, the analytic form of the superpotential W can be found [25]. However, for $p > 2$, W cannot be expressed in a closed analytic form [25].

On the other hand, Comtet *et al* [16] have modified the usual WKB approximation to obtain the SWKB approximation, where the quantization rule is given explicitly by

$$\int_{x_1}^{x_2} \sqrt{\tilde{E}_n^{(1)} - W^2(x)}\, \mathrm{d}x = n\pi, \tag{1.11}$$

for $n = 0, 1, 2, \ldots$. Here $\tilde{E}_n$ is SWKB energy of the nth state, and x_1, x_2 are turning points defined by $\tilde{E}_n^{(1)} = W^2(x_1) = W^2(x_2)$. The SWKB formula can be obtained from the traditional WKB quantization rule [28],

$$\int_{x_1'}^{x_2'} \sqrt{\tilde{E}_n^{(1)} - V_1(x)}\, \mathrm{d}x = \left(n + \frac{1}{2}\right)\pi, \tag{1.12}$$

where x_1', x_2' are the 'turning points' defined by $V(x_1') = V(x_2') = \tilde{E}_n$. The lowest order SWKB formula (1.11) is then the leading term of an expansion of the WKB formula (1.12) in power series of $\hbar$ [16].

The SWKB approximation is generally deemed to outperform the conventional WKB method for the following reasons [19, 29]. First, the SWKB method reduces to the conventional WKB method in the large-n limit. Therefore, the SWKB method is expected to produce accurate results for high-lying excited states as the conventional WKB method does. Second, the SWKB quantization rule yields the correct energy $E_0^{(1)} = 0$ for the ground state, in sharp contrast to the poor performance of the conventional WKB method for low-lying states. More interestingly, the SWKB approximation (1.11) was found to give the exact result for all TSIPs [16, 17]. Such unexpected exactness of SWKB for TSIPs has attracted the interest of researchers in the field of SQM and there have been a lot of attempts aiming at its explanation [16–18, 30–34]. For example, Dutt *et al* [17] claimed that the exactness stems from the level degeneracy between the partner systems. However, their argument failed to explain why (1.11) does not yield exact results for SSIPs where level degeneracy still prevails [30]. More recently, Milczarski and Giller [34] and Cheng *et al* [20] have attributed the exactness of the SWKB formula for TSIPs to the analytic property and the generic behaviour of the superpotential W, respectively.

The main objective of the present paper is to study the performance of the SWKB approximation for CSIPs. Specifically, we first study period-2 CSIPs, whose superpotential W can be obtained analytically. We find that the SWKB quantization formula can yield highly accurate numerical results for either the even or the odd states, but not both. To substantiate our numerical results, we evaluate the SWKB integral analytically and succeed in showing that the SWKB formula can yield *exact* eigenenergy for either the even states or the odd states, depending on the parameters characterizing the superpotential. To the best of our knowledge, such an alternating behaviour in the quality of SWKB approximation has not yet been demonstrated in other systems. For CSIPs with periods greater than 2, we further note that the accuracy of the SWKB formula also demonstrate similar alternating patterns. However, the SWKB quantization formula fails to produce highly accurate results even in the high-energy limit owing to the rapid oscillations in the potential at large distances.

The organization of our paper is as follows. Following a brief review on the construction of CSIPs in section 2, we discuss the properties of period-2 CSIPs in section 3 and establish explicitly the exactness of the SWKB approximation for the even (or odd) states of period-2 CSIPs in section 4. In section 5, we study the alternate exactness of the SWKB approximation for period-2 CSIPs and show that such intriguing behaviour actually stems from the exactness of the SWKB approximation for TSIPs. We then generalize our study to CSIPs with greater periods in section 6 and end this paper with a brief discussion in section 7.

2. Construction of CSIPs

For CSIPs, the parameters a_1 and a_2 in the shape invariant condition (1.8) are related by a cyclic function which reduces to an identity map after p iterations. By using the cyclic

condition that $W(x, a_{p+1}) = W(x, a_1)$ and $R(a_{p+1}) = R(a_1)$, the shape invariant condition for CSIPs system can be written as p-coupled ordinary differential equations [25],

$$W_k^2 + W_k' = W_{k+1}^2 - W_{k+1}' + \omega_k, \tag{2.1}$$

where $W_k = W(x, a_k)$ and $\omega_k = R(a_k) > 0$ for $p \geqslant k \geqslant 1$. The superpotential W_k can be obtained from these p differential equations, either analytically or numerically [25]. Furthermore, equation (1.9) also gives the exact energies of Hamiltonian $\hat{H}_1$,

$$E_n = j\Omega_p + \sum_{k=1}^{n-j} \omega_k, \tag{2.2}$$

where jp is the highest multiple of p smaller than n, $\Omega_p = \sum_{k=1}^{p} \omega_k$, and hereafter $E_n = E_n^{(1)}$. In the subsequent discussion, we will construct the superpotential W_k from (2.1) and evaluate the SWKB integral (1.11) for CSIPs with $p = 2, 3, 4, 5$. The energies obtained from the SWKB approximation are then compared with the exact energies given by (2.2).

3. Period-2 CSIPs

To gain more insight of the problem, we first study the simplest nontrivial CSIP, namely the period-2 CSIP. For period-2 CSIPs, equation (2.1) contains only two ordinary differential equations

$$W_1^2 + W_1' = W_2^2 - W_2' + \omega_1; \tag{3.1}$$

$$W_2^2 + W_2' = W_1^2 - W_1' + \omega_2. \tag{3.2}$$

These two equations can be solved analytically and the superpotentials are given by

$$W_1(x) = \frac{(\omega_1 + \omega_2)x}{4} + \frac{(\omega_1 - \omega_2)}{2(\omega_1 + \omega_2)x}, \tag{3.3}$$

$$W_2(x) = \frac{(\omega_1 + \omega_2)x}{4} + \frac{(\omega_2 - \omega_1)}{2(\omega_1 + \omega_2)x}. \tag{3.4}$$

It is worthy of note that the superpotentials W_1 and W_2 have the same functional form except with the indices 1, 2 switched. Moreover, it is beneficial to consider $(\omega_1 - \omega_2)/[2(\omega_1 + \omega_2)]$ and $(\omega_2 - \omega_1)/[2(\omega_1 + \omega_2)]$ as the parameters a_1 and a_2, respectively. It is then obvious that the function f in (1.10) is simply given by $f(a_1) = -a_1$, which is of course a period-2 cyclic function.

From W_1 and W_2 obtained above, the corresponding potentials V_1 and V_2 can be readily found,

$$V_1(x) = \frac{(\omega_1 + \omega_2)^2 x^2}{16} - \frac{\omega_2}{2} + \frac{(3\omega_1 + \omega_2)(\omega_1 - \omega_2)}{4(\omega_1 + \omega_2)^2 x^2}, \tag{3.5}$$

$$V_2(x) = \frac{(\omega_1 + \omega_2)^2 x^2}{16} + \frac{\omega_1}{2} + \frac{(\omega_1 + 3\omega_2)(-\omega_1 + \omega_2)}{4(\omega_1 + \omega_2)^2 x^2}. \tag{3.6}$$

For the purpose of illustration, the superpotential W_1 and potential V_1 are shown in figure 1 for the cases with (i) $\omega_1 = 0.7, \omega_2 = 0.3$ (dotted line); and (ii) $\omega_1 = 0.3, \omega_2 = 0.7$ (solid line).

From the explicit expression of these superpotentials and potentials, one can easily see that period-2 CSIPs are in fact similar to three-dimensional harmonic oscillators with fractional

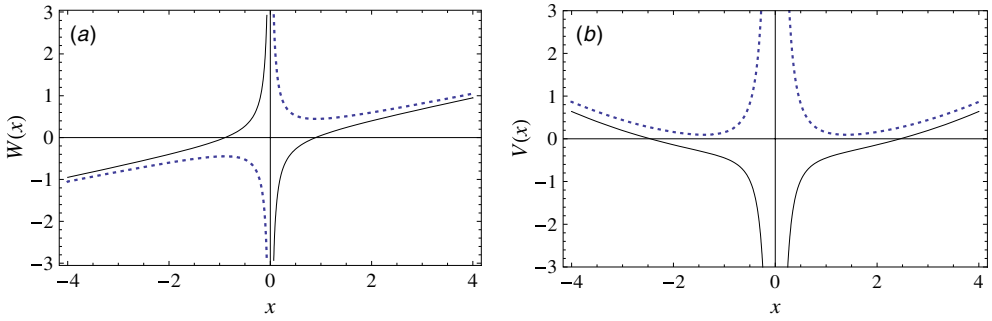

Figure 1. (*a*) The superpotential W_1 and (*b*) the potential V_1 are shown as functions of x for period-2 CSIPs with (i) $\omega_1 = 0.7$, $\omega_2 = 0.3$ (dotted line); and (ii) $\omega_1 = 0.3$, $\omega_2 = 0.7$ (solid line). These two cases are a pair of supersymmetric partner potentials.

angular momentum $l = (\omega_1 - \omega_2)/[2(\omega_1 + \omega_2)]$. It is well known that three-dimensional harmonic oscillators are a kind of TSIP (see, e.g., [6]). Therefore, period-2 CSIPs can also be treated as TSIPs. We will see later in this paper that it is this salient feature of period-2 CSIPs leading to the alternate exactness of the SWKB approximation. The dual role of period-2 CSIPs can be understood as follows. For a period-2 CSIP with a potential $V_1(x)$ carrying a fractional angular momentum l and hence a centrifugal barrier $l(l+1)/x^2$, the shape invariant function is given by $f(l) = -l$. Then, the angular momentum of its partner potential $V_2(x)$ is $-l$ and the centrifugal barrier in $V_2(x)$ is $-l(-l+1)/x^2$, or equivalently $l(l-1)/x^2$. Hence, $V_1(x)$ and $V_2(x)$ are characterized by angular momenta l and $l-1$, which are of the translational invariant type. As a result, period-2 CSIPs are also a special kind of TSIP.

Meanwhile, the exact energy spectrum is given by (1.9)

$$E_n = \begin{cases} n(\omega_1 + \omega_2)/2 & \text{if} \quad n \text{ is even;} \\ (n-1)(\omega_1 + \omega_2)/2 + \omega_1 & \text{if} \quad n \text{ is odd.} \end{cases} \tag{3.7}$$

Besides, the corresponding eigenfunction $\psi_n(x) = \psi_n^{(1)}(x)$ are given by

$$\psi_{2m+j}(x) = x^{j-\Delta/2} \exp(-\omega x^2/4) L_n^{(-\Delta/2+j-1/2)}(\omega x^2/2), \tag{3.8}$$

where $m = 0, 1, 2, 3, \ldots$, $j = 0, 1$, $L_n^{(\alpha)}(z)$ is the standard generalized Laguerre polynomial [35], and

$$\Delta(\omega_1, \omega_2, j) = (-1)^j \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}, \tag{3.9}$$

$$\omega = \frac{\omega_1 + \omega_2}{2}. \tag{3.10}$$

Before delving into the application of the SWKB approximation to period-2 CSIPs, it is worthwhile to note that period-2 (and other even period) CSIPs are singular at the origin $x = 0$, where the potential is proportional to $1/x^2$. It is well known that for an inverse-square singular potential $V_s(x) = \lambda/x^2$, the behaviour of the wavefunction at the origin depends crucially on the strength λ (see [4] and references therein). If $\lambda > 3/4$, the potential is said to have a 'strong' singularity there. The two intervals $x > 0$ and $x < 0$ become disjoint and the particle cannot tunnel through the point $x = 0$. On the other hand, if $-1/4 < \lambda < 3/4$, the potential then has a singularity of 'intermediate' strength. Such an 'intermediate' singularity is not strong enough to make the wavefunction there vanish and the particle can tunnel through the origin. Hence, the two intervals $x > 0$ and $x < 0$ are connected and one has to solve the

Schrödinger equation in the full range $-\infty < x < \infty$ [4, 26]. It is easy to see from (3.5) that period-2 CSIPs have an intermediate singularity at the origin. As a consequence, one has to consider the whole range $-\infty < x < \infty$ when solving the Schrödinger equation for period-2 (and other even-period) CSIPs. Similarly, the SWKB approximation should be applied to the whole range $-\infty < x < \infty$.

4. SWKB approximation for period-2 CSIPs

We are now going to compare this exact spectrum in (3.7) with that obtained from the SWKB formula for period-2 CSIPs. Without loss of generality, in the following we evaluate the SWKB integral for a superpotential given by

$$W = W_1 = Ax + \frac{B}{x}, \tag{4.1}$$

where

$$A = \frac{\omega_1 + \omega_2}{4} \tag{4.2}$$

is always positive, and

$$B = \frac{\omega_1 - \omega_2}{2(\omega_1 + \omega_2)} \tag{4.3}$$

can be positive (negative) if $\omega_1 - \omega_2 > 0$ ($\omega_1 - \omega_2 < 0$). It is worthy of remark that if $B > 0$, W^2 attains a non-vanishing minimum of $(\omega_1 - \omega_2)/2$ at $x = [2(\omega_1 - \omega_2)]^{1/2}/(\omega_1 + \omega_2)$. However, if $B < 0$, the minimum of W^2 is zero.

Direct substitution of (4.1) into the SWKB formula (1.11) leads to the quantization rule

$$I_n = \left(\int_{x_a}^{x_b} + \int_{x_c}^{x_d}\right) \mathrm{d}x \sqrt{\tilde{E}_n - \left(A^2x^2 + 2AB + \frac{B^2}{x^2}\right)} = n\pi, \tag{4.4}$$

where $\tilde{E}_n$ is the nth eigenenergy obtained from the SWKB formula and

$$x_{a,b,c,d} = \frac{\pm\sqrt{\tilde{E}_n} \pm \sqrt{\tilde{E}_n - 4AB}}{2A} \tag{4.5}$$

are the four turning points defined by $W^2 = \tilde{E}_n$ with $x_a < x_b < x_c < x_d$. It is worthy of note that owing to the singular nature of W^2 at $x = 0$, the integration in (4.4) contains two disconnected parts.

We make use of the SWKB quantization rule (1.11) to determine $\tilde{E}_n$ and the numerical results for four typical sets of ω_1 and ω_2 are shown in table 1. It is surprising to see that for all these cases the SWKB energy $\tilde{E}_n$ is, within acceptable numerical errors, identical to the exact energy E_n if either (i) $\omega_1 > \omega_2$ and $n = 2m + 1$; or (ii) $\omega_1 < \omega_2$ and $n = 2m$, where $m = 0, 1, 2, 3, \ldots$. This intriguing discovery strongly suggests that the SWKB quantization rule (1.11) is exact in an alternate manner for period-2 CSIPs. In addition, contrary to the conventional belief that the SWKB approximation can lead to exact ground-state energy, which is zero, the SWKB approximation yields nonzero ground-state energy if $\omega_1 > \omega_2$. In the following discussion, we will establish such exactness of the SWKB quantization rule analytically.

The integral in (4.4) can be evaluated analytically with standard methods in residue calculus by considering the integration variable x as a complex variable z. The integrand in (4.4),

$$J(z) \equiv \sqrt{\tilde{E}_n - \left(A^2z^2 + 2AB + \frac{B^2}{z^2}\right)}, \tag{4.6}$$

Table 1. The table shows the numerical values of $\tilde{E}_n$ obtained from the SWKB quantization rule (1.11) for period-2 CSIPs with four typical sets of ω_1 and ω_2. $\tilde{E}_n$ is, within acceptable numerical errors, identical to the exact energy E_n if either (i) $\omega_1 > \omega_2$ and $n = 2m + 1$; or (ii) $\omega_1 < \omega_2$ and $n = 2m$, where $m = 0, 1, 2, 3, \ldots$.

ω_1	ω_2	n	E_n	$\tilde{E}_n$	$E_n - \tilde{E}_n$	$\tilde{E}_n - \tilde{E}_{n-1}$
0.7	0.3	0	0.000 00	0.200 00	−0.200 00	NA
		1	0.700 00	0.700 00	0.000 00	0.500 00
		2	1.000 00	1.200 00	−0.200 00	0.500 00
		3	1.700 00	1.700 00	0.000 00	0.500 00
		4	2.000 00	2.199 99	−0.199 99	0.499 98
		5	2.700 00	2.700 00	0.000 00	0.500 01
		10	5.000 00	5.19999	−0.19999	0.49998
		15	7.700 00	7.700 00	0.000 00	0.500 02
		20	10.000 00	10.200 04	−0.200 04	0.500 02
0.3	0.7	0	0.000 00	0.000 00	0.000 00	NA
		1	0.300 00	0.500 00	−0.200 00	0.500 00
		2	1.000 00	1.000 00	0.000 00	0.500 00
		3	1.300 00	1.499 99	−0.199 99	0.500 00
		4	2.000 00	2.000 00	0.000 00	0.500 00
		5	2.300 00	2.500 00	−0.200 00	0.500 01
		10	5.000 00	4.999 99	0.000 01	0.500 01
		15	7.300 00	7.500 02	−0.199 98	0.499 99
		20	10.000 00	9.999 98	0.000 02	0.499 98
0.6	0.4	0	0.000 00	0.100 00	−0.100 00	NA
		1	0.600 00	0.600 00	0.000 00	0.500 00
		2	1.000 00	1.100 00	−0.100 00	0.500 00
		3	1.600 00	1.600 00	0.000 00	0.500 00
		4	2.000 00	2.099 99	−0.099 99	0.499 99
		5	2.600 00	2.600 00	0.000 00	0.500 01
		10	5.000 00	5.099 99	−0.099 99	0.499 98
		15	7.600 00	7.599 99	0.000 01	0.500 00
		20	10.000 00	10.099 98	−0.099 98	0.499 98
0.4	0.6	0	0.000 00	0.000 00	0.000 00	NA
		1	0.400 00	0.500 00	−0.100 00	0.500 00
		2	1.000 00	1.000 00	0.000 00	0.500 00
		3	1.400 00	1.500 01	−0.100 00	0.500 01
		4	2.000 00	1.999 99	0.000 01	0.499 98
		5	2.400 00	2.499 99	−0.099 99	0.500 00
		10	5.000 00	5.000 01	−0.000 01	0.500 01
		15	7.400 00	7.500 00	−0.100 00	0.499 96
		20	10.000 00	9.999 99	0.000 01	0.500 00

has two cuts along the real z-axis, going respectively from x_a to x_b and from x_c to x_d, and a singular point at $z = 0$ as well. It is then readily shown that

$$I_n = -\frac{1}{2}\oint_{C_1+C_2} J(z)\,\mathrm{d}z, \tag{4.7}$$

where C_1 (C_2) is the contour enclosing the cut from x_a to x_b (x_c to x_d) in the counterclockwise direction (see figure 2).

On the other hand, as there is neither pole nor cut in the complex plane outside C_0, C_1, C_2, where C_0 is the contour enclosing the singular point at $z = 0$, the corresponding integral along

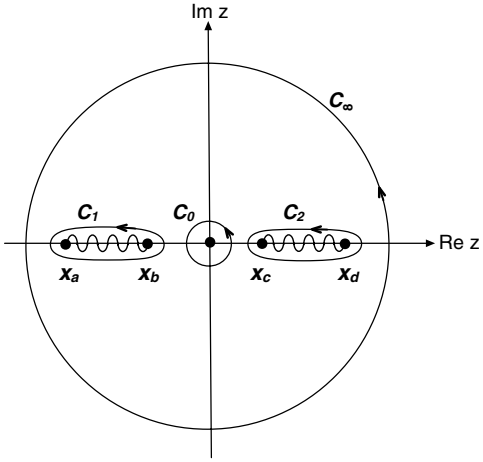

Figure 2. The figure shows the integration contours in the complex z-plane, which are used to calculate the SWKB integral and hence $\tilde{E}_n$ for CSIPs with $p = 2$.

an infinitely large contour C_∞ enclosing the point $z = \infty$ is given by

$$\oint_{C_\infty} J(z)\,\mathrm{d}z = \left(\oint_{C_0} + \oint_{C_1} + \oint_{C_2}\right) J(z)\,\mathrm{d}z. \tag{4.8}$$

Furthermore, from standard methods in residue calculus, the integrals along C_0 and C_∞ can be obtained,

$$\oint_{C_0} J(z)\,\mathrm{d}z = -2\pi|B|, \tag{4.9}$$

$$\oint_{C_\infty} J(z)\,\mathrm{d}z = -\frac{\pi(\tilde{E}_n - 2AB)}{A}. \tag{4.10}$$

Following directly from the SWKB quantization condition (4.4), equations (4.7), (4.8), (4.9) and (4.10), the energy of the nth eigenstate under the SWKB approximation is given by

$$\tilde{E}_n = 2nA + 2A|B| + 2AB. \tag{4.11}$$

It is worthy of note that the SWKB eigenenergy $\tilde{E}_n$ is non-analytic in B due to the presence of $|B|$ in the RHS of (4.11). In the following discussion, we will see that whether $\tilde{E}_n$ agrees with the exact value of E_n in (3.7) in fact depends on the sign of B.

To study the question of the exactness of the SWKB formula, we express the parameters A, B in terms of ω_1 and ω_2 and substitute them back into (4.4). We firstly consider the case with $\omega_1 > \omega_2$. Since B is positive in this case, the energy spectrum generated by the SWKB formula is

$$\tilde{E}_n = 2nA + 4AB = \frac{(n-1)(\omega_1 + \omega_2)}{2} + \omega_1. \tag{4.12}$$

When comparing $\tilde{E}_n$ with the exact energy given by (3.7), it is interesting to find that SWKB quantization successfully reproduces exact energies for all odd states with $n = 1, 3, 5, \ldots$, but not for the even states. Most interestingly, as mentioned above, in this case SWKB is inexact even for the ground state. Instead, it is easy to see that $\tilde{E}_0 = (\omega_1 - \omega_2)/2$ from the SWKB formula (4.12). Alternately, the result can be understood because the minimum value of W^2 is $(\omega_1 - \omega_2)/2$, which is equal to $\tilde{E}_0$ by (1.11) with $n = 0$.

Second, we consider the case with $\omega_1 < \omega_2$, implying that B is negative, and thus the spectrum generated by SWKB formula is

$$\tilde{E}_n = 2nA = \frac{n}{2}(\omega_1 + \omega_2). \tag{4.13}$$

In this case, the SWKB approximation gives exact values of energies for all even states with $n = 0, 2, 4, \ldots$, but not for the odd states.

From the above calculations, it is surprising to find that the SWKB formula for period-2 CSIPs is neither completely exact (as for TSIPs) nor completely inexact (as for SSIPs). Instead, for period-2 CSIPs the SWKB formula is exact for either all the even states or all the odd states, depending merely on the value of $\omega_1 - \omega_2$.

5. Period-2 CSIPs, TSIPs and broken SUSY

Here we analyse in detail the alternate exactness of the SWKB approximation for the period-2 CSIP. First, as the SWKB approximation is exact for all known TSIPs, and period-2 CSIPs can also be considered as a kind of TSIP, it is plausible that the alternate exactness discovered here for period-2 CSIPs follows directly from the exactness of the SWKB approximation for TSIPs. In addition, we also note that such intriguing phenomenon is closely related to broken SUSY (BSUSY), where the SWKB formula (1.11) has to be modified [20, 36–39]. In SQM, BSUSY refers to supersymmetric partner potentials with both $E_0^{(1)} \neq 0$ and $E_0^{(2)} \neq 0$. As a result, each eigenenergy of $\hat{H}_1$ is identical to the corresponding one of $\hat{H}_2$. In systems where BSUSY prevails, the SWKB quantization formula needs to be modified as follows [20, 38, 39]:

$$\int_{x_1}^{x_2} \sqrt{E_n^{(1)} - W^2(x)}\, \mathrm{d}x = \left(m + \frac{1}{2}\right)\pi, \tag{5.1}$$

for $m = 0, 1, 2, \ldots$. As mentioned above, x_1, x_2 are turning points defined by $E_n^{(1)} = W^2(x_1) = W^2(x_2)$. The extra $1/2$ in the RHS of (5.1) arises from the difference in the asymptotic behaviour of the superpotential $W(x)$ for systems with unbroken and broken SUSY [39]. Generally speaking, the signs of $W(x)$ at the two spatial boundaries are opposite (the same) for systems with unbroken (broken) SUSY. It is such difference that gives rise to the two different SWKB quantization formulae, equations (1.11) and (5.1) [39].

To see the relation between BSUSY and the alternate exactness of the SWKB formula for period-2 CSIPs, we first study specifically the case with $\omega_1 > \omega_2$. By virtue of the reflection symmetry of the potential about the point $x = 0$, we can consider the Schrödinger wave equation in the half-line where $0 < x < \infty$. Hence, the two spatial boundary points are $x = 0$ and $x = \infty$. Following directly from (3.3), the limits of the superpotential W_1 at these boundaries are both positive (see figure 1) and the system is thus characterized by BSUSY. In particular, the ground state $\psi_0^{(1)}$ of V_1 is not a physical state because the momentum operator $-\mathrm{i}\mathrm{d}/\mathrm{d}x$ is no longer Hermitian for such a state, which can be shown in a straightforward way from (3.8) [36]. Besides, the energy of $\psi_0^{(1)}$ is zero despite the fact that the potential V_1 is greater than zero everywhere. The non-Hermitian property of the momentum operator is of course the culprit of this unphysical result [36]. Once $\psi_0^{(1)}$ and $E_0^{(1)}$ are excluded from the physical solutions of the Schrödinger wave equation for V_1, the spectra of V_1 and V_2 become completely degenerate, which is a standard characteristic of BSUSY.

The SWKB quantization rule for systems with BSUSY is given by (5.1), instead of (1.11). Using (5.1) and by reflection symmetry of V_1, we find that

$$\left(\int_{x_a}^{x_b} + \int_{x_c}^{x_d}\right) \sqrt{E_n^{(1)} - W^2(x)}\, \mathrm{d}x = (2m + 1)\,\pi, \tag{5.2}$$

where $m = 0, 1, 2, \ldots$. As mentioned above, period-2 CSIPs are also a kind of TSIPs. Since the exactness of the SWKB quantization rule for TSIPs with BSUSY is well established [20, 38, 39], the alternate exactness of the SWKB quantization rule for period-2 CSIPs follows directly from (5.2). By comparing (1.11) and (5.2), one can readily see that the former is exact for all the odd states with $n = 1, 3, 5, \ldots$, but not for the even states.

Second, we consider the other case with $\omega_1 < \omega_2$. Proceeding along similar arguments, we find that SUSY of the system is unbroken as the limits of the superpotential W_1 at the two spatial boundaries are of opposite signs (see figure 1). Therefore, the standard SWKB formula for unbroken supersymmetry should be applied, yielding the result

$$\left(\int_{x_a}^{x_b} + \int_{x_c}^{x_d}\right)\sqrt{E_n^{(1)} - W^2(x)}\,\mathrm{d}x = 2m\pi, \tag{5.3}$$

where $m = 0, 1, 2, \ldots$. It is then obvious that (1.11) is exact for all the even states with $n = 0, 2, 4, \ldots$, but not for the odd states.

6. SWKB approximation for other CSIPs

In this section, we further studies to other CSIPs. However, for CSIPs with period $p \geqslant 3$, the superpotential $W(x)$ cannot be obtained analytically. We consider, as an example, period-3 CSIPs, whose superpotentials are governed by three coupled ordinary differential equations [25]

$$\begin{aligned} W_1' &= W_2^2 - W_3^2 + \tfrac{1}{2}(\omega_1 - \omega_2 + \omega_3), \\ W_2' &= W_3^2 - W_1^2 + \tfrac{1}{2}(\omega_2 - \omega_3 + \omega_1), \\ W_3' &= W_1^2 - W_2^2 + \tfrac{1}{2}(\omega_3 - \omega_1 + \omega_2). \end{aligned} \tag{6.1}$$

To our knowledge, there is no closed form solution to (6.1). In the region where $|x| \ll 1$, the superpotential W_1 can be expanded as follows [25]:

$$W_1(x) = \frac{x}{2}(\omega_1 - \omega_2 + \omega_3) + \frac{x^3}{3}\omega_2(\omega_1 - \omega_3) + \cdots. \tag{6.2}$$

Unlike the case of period-2 CSIPs, W_1 is finite and well behaved at small x. In general, superpotentials of CSIPs with odd periods are all well-behaved functions at small x and they vanish at the point $x = 0$ [25]. For intermediate and large x, we have to solve these differential equations numerically with the standard Runge–Kutta method. In figure 3, we show the numerical results of both $W_1(x)$ and $V_1(x)$ for the parameters $\omega_1 = 0.3$, $\omega_2 = 0.4$, $\omega_3 = 0.7$.

After numerically evaluating the superpotentials, we obtain the SWKB energy spectrum by numerical integration and root-search schemes. The numerical result are then compared with the exact spectrum given by the shape invariant condition (1.9),

$$E_n = \begin{cases} n(\omega_1 + \omega_2 + \omega_3)/3 & \text{for} \quad n = 3m, \\ (n-1)(\omega_1 + \omega_2 + \omega_3)/3 + \omega_1 & \text{for} \quad n = 3m+1, \\ (n-2)(\omega_1 + \omega_2 + \omega_3)/3 + \omega_1 + \omega_2 & \text{for} \quad n = 3m+2, \end{cases} \tag{6.3}$$

where $m = 0, 1, 2, \ldots$.

The accuracy of the SWKB formula for period-3 CSIPs can be observed from the data presented as table 2. From the table, it is then obvious that for period-3 CSIPs the SWKB formula is neither exact nor alternately exact as in the case of period-2 CSIPs. In particular, in stark contrast to conventional belief, the accuracies of the SWKB results do not improve much with increasing n. On the other hand, the difference between the exact and the SWKB spectra,

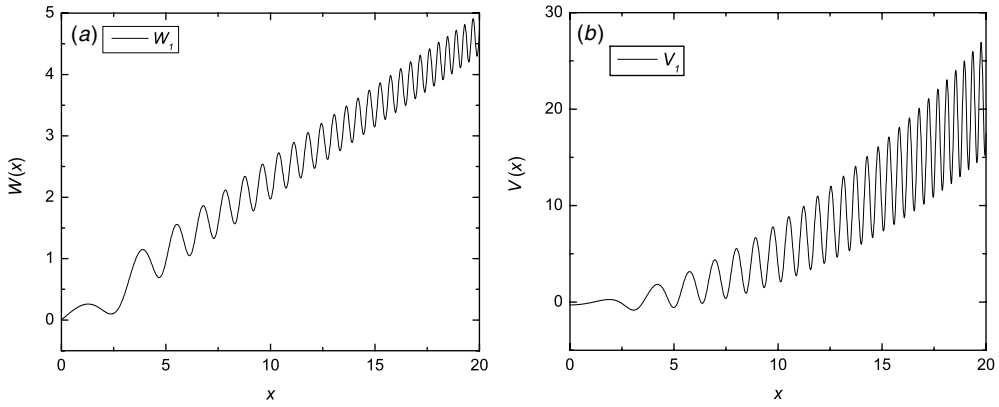

Figure 3. (*a*) The superpotential W_1 and (*b*) the potential V_1 of a period-3 CSIP with parameters $\omega_1 = 0.3, \omega_2 = 0.4, \omega_3 = 0.7$ are shown as functions of x.

$E_n - \tilde{E}_n$, also demonstrates a quasi-periodic variation for every 3 levels, which becomes more regular for large n. More interestingly, we find that such a quasi-periodic variation in $E_n - \tilde{E}_n$ persists in other CSIPs. As shown in tables 3 and 4, where E_n and $\tilde{E}_n$ of period-4 and period-5 CSIPs are tabulated, $E_n - \tilde{E}_n$ manifests similar quasi-periodic variation for every 4 and 5 levels, respectively. In general, for a period-p CSIP, the difference $E_n - \tilde{E}_n$ varies quasi-periodically for every p levels. Besides, the accuracies of the energy levels obtained from the SWKB approximation also do not improve with increasing n.

Here we provide a physical argument for such a intriguing quasi-periodical behaviour for a general period-p CSIP ($p > 2$). Sukhatme *et al* [25] have shown that at large x the superpotential $W(x)$ assume the form

$$W(x) = \frac{\Omega_p x}{2p} + \sum_j b_j \sin(q_j x^2 + \phi_j) + O\left(\frac{1}{x}\right), \tag{6.4}$$

where $j = 1, 2, 3, \ldots$ and is bounded by $p/2$, $q_j = (\Omega_p/2p)\tan(j\pi/p)$, and b_j and ϕ_j are constants to be determined from (2.1). As a result, $W(x)$ oscillates rapidly at large x. This point is clearly demonstrated in figures 3–5, where the superpotentials and the potentials of period-3, 5 and 4 CSIPs are plotted against x, respectively.

In order to find $\tilde{E}_n$, one has to evaluate the integral in (1.11). In particular, if one is interested in the large-n regime, the integral in (1.11) is dominated by the large-x limit of $W(x)$. Taking into consideration the rapidly oscillating asymptotic behaviour of $W(x)$ there, we can replace $W(x)$ by its spatial average $\overline{W(x)}$, and from (6.4), $\overline{W(x)}$ is equal to $\Omega_p x/(2p)$. Making use of this approximation and the SWKB formula, we get

$$\int \sqrt{\tilde{E}_n - \left(\frac{\Omega_p x}{2p}\right)^2}\, \mathrm{d}x \approx n\pi, \tag{6.5}$$

from which the SWKB approximate energies for high excited states are

$$\tilde{E}_n \approx \frac{n\Omega_p}{p}. \tag{6.6}$$

Hence the SWKB formula approximately gives an evenly spaced spectrum for large n. In fact, this point can be easily observed and verified from tables 2–4. Comparing $\tilde{E}_n$ shown in

Table 2. A table showing the exact energy E_n and the SWKB energy $\tilde{E}_n$ for period-3 CSIPs with different values of ω_1, ω_2 and ω_3.

ω_1	ω_2	ω_3	n	E_n	$\tilde{E}_n$	$E_n - \tilde{E}_n$	$\tilde{E}_n - \tilde{E}_{n-1}$
0.3	0.4	0.7	0	0.000 00	0.000 00	0.000 00	NA
			1	0.300 00	0.300 07	−0.000 07	0.300 07
			2	0.700 00	0.797 67	−0.097 67	0.497 61
			3	1.400 00	1.277 88	0.122 12	0.480 20
			4	1.700 00	1.731 35	−0.031 35	0.453 47
			5	2.100 00	2.206 96	−0.106 96	0.475 61
			6	2.800 00	2.658 77	0.141 23	0.451 82
			7	3.100 00	3.141 80	−0.041 80	0.483 02
			8	3.500 00	3.592 91	−0.092 91	0.451 11
			9	4.200 00	4.068 15	0.131 85	0.475 25
			10	4.500 00	4.532 00	−0.032 00	0.463 84
			11	4.900 00	4.998 51	−0.098 51	0.466 51
			12	5.600 00	5.468 46	0.131 54	0.469 95
			13	5.900 00	5.932 61	−0.032 61	0.464 15
			14	6.300 00	6.401 18	−0.101 18	0.468 57
			15	7.000 00	6.868 42	0.131 58	0.467 24
			16	7.300 00	7.334 95	−0.034 95	0.466 53
			17	7.700 00	7.798 49	−0.098 49	0.463 54
			18	8.400 00	8.270 19	0.129 81	0.471 70
			19	8.700 00	8.730 74	−0.030 74	0.460 55
			20	9.100 00	9.203 71	−0.103 71	0.472 98
0.3	0.7	0.4	0	0.000 00	0.000 00	0.000 00	NA
			1	0.300 00	0.409 88	−0.109 88	0.409 88
			2	1.000 00	0.921 63	0.078 37	0.511 75
			3	1.400 00	1.349 63	0.050 37	0.428 00
			4	1.700 00	1.836 09	−0.136 09	0.486 45
			5	2.400 00	2.301 90	0.098 10	0.465 81
			6	2.800 00	2.761 78	0.038 22	0.459 88
			7	3.100 00	3.239 73	−0.139 73	0.477 95
			8	3.800 00	3.697 32	0.102 68	0.457 59
			9	4.200 00	4.169 80	0.030 20	0.472 48
			10	4.500 00	4.630 75	−0.13075	0.46095
			11	5.200 00	5.103 53	0.096 47	0.472 79
			12	5.600 00	5.561 55	0.038 45	0.458 01
			13	5.900 00	6.038 85	−0.138 85	0.477 30
			14	6.600 00	6.495 49	0.104 51	0.456 64
			15	7.000 00	6.968 18	0.031 82	0.472 69
			16	7.300 00	7.431 55	−0.131 55	0.463 37
			17	8.000 00	7.900 08	0.099 92	0.468 53
			18	8.400 00	8.365 32	0.034 68	0.465 24
			19	8.700 00	8.833 95	−0.133 95	0.468 63
			20	9.400 00	9.297 13	0.102 87	0.463 18
0.1	0.8	0.9	0	0.000 00	0.000 00	0.000 00	NA
			1	0.100 00	0.419 18	−0.319 18	0.419 18
			2	0.900 00	0.915 78	−0.01578	0.49660
			3	1.800 00	1.525 96	0.27404	0.61018
			4	1.900 00	2.153 92	−0.25392	0.62797
			5	2.700 00	2.759 28	−0.059 28	0.605 36
			6	3.600 00	3.323 11	0.276 89	0.563 84
			7	3.700 00	3.933 68	−0.233 68	0.610 56

Table 2. (Continued).

ω_1	ω_2	ω_3	n	E_n	$\tilde{E}_n$	$E_n - \tilde{E}_n$	$\tilde{E}_n - \tilde{E}_{n-1}$
			8	4.500 00	4.546 14	−0.046 14	0.612 47
			9	5.400 00	5.129 58	0.270 42	0.583 43
			10	5.500 00	5.733 75	−0.233 75	0.604 17
			11	6.300 00	6.340 40	−0.040 40	0.606 65
			12	7.200 00	6.929 82	0.270 18	0.589 42
			13	7.300 00	7.537 12	−0.237 12	0.607 30
			14	8.100 00	8.136 61	−0.036 61	0.599 48
			15	9.000 00	8.731 00	0.269 00	0.594 39
			16	9.100 00	9.339 03	−0.239 03	0.608 03
			17	9.900 00	9.930 49	−0.030 49	0.591 46
			18	10.800 00	10.533 67	0.266 33	0.603 18
			19	10.900 00	11.137 03	−0.237 03	0.603 35
			20	11.700 00	11.732 34	−0.032 34	0.595 31

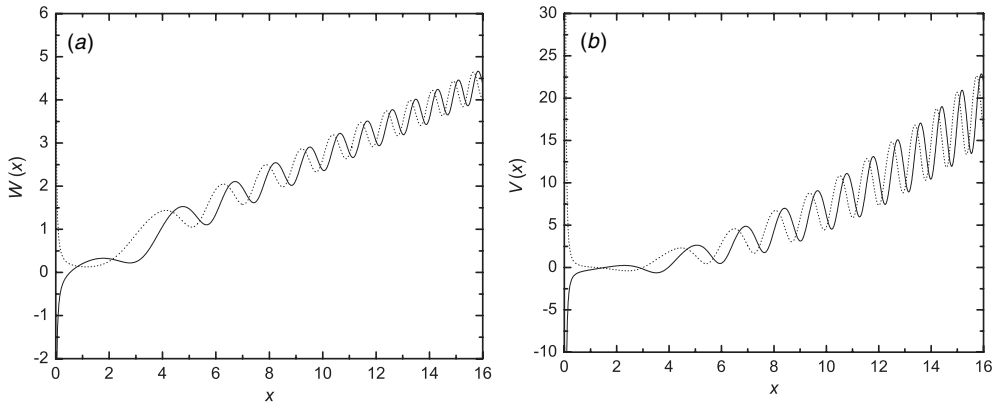

Figure 4. (*a*) The superpotential W_1 and (*b*) the potential V_1 are shown as functions of x for two period-4 CSIPs, respectively, with the parameters $(\omega_1, \omega_2, \omega_3, \omega_4)$ given by $(0.3, 0.4, 0.6, 0.9)$ for the solid line, and $(0.4, 0.6, 0.9, 0.3)$ for the dotted line. These two cases are a pair of supersymmetric partner potentials.

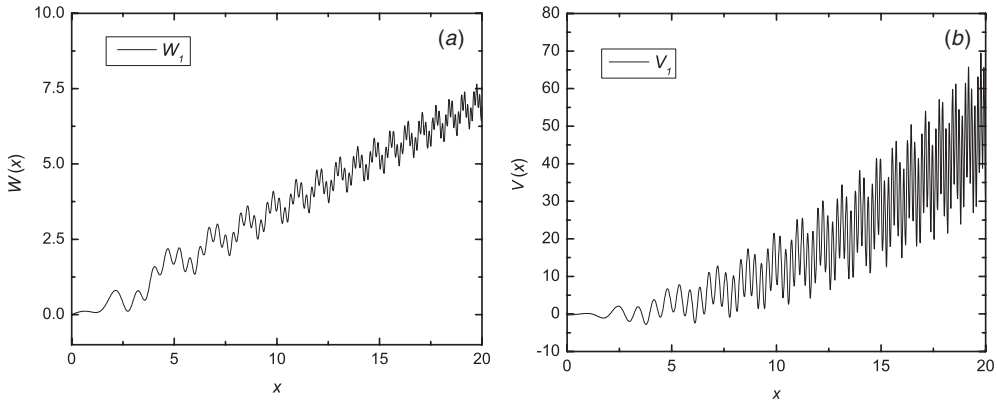

Figure 5. (*a*) The superpotential W_1 and (*b*) the potential V_1 of a period-5 CSIP with parameters $\omega_1 = 0.4, \omega_2 = 0.5, \omega_3 = 0.55, \omega_4 = 1.0$, and $\omega_5 = 1.1$ are shown as functions of x.

Table 3. A table showing the exact energy E_n and the SWKB energy $\tilde{E}_n$ for period-4 CSIPs with different values of $\omega_1, \omega_2, \omega_3$ and ω_4.

ω_1	ω_2	ω_3	ω_4	n	E_n	$\tilde{E}_n$	$E_n - \tilde{E}_n$	$\tilde{E}_n - \tilde{E}_{n-1}$
0.3	0.4	0.6	0.9	0	0.0000	0.0000	0.0000	NA
				1	0.3000	0.2999	0.0001	0.2999
				2	0.7000	0.8382	−0.1382	0.5382
				3	1.3000	1.4690	−0.1690	0.6309
				4	2.2000	2.0324	0.1676	0.5634
				5	2.5000	2.5401	−0.0401	0.5077
				6	2.9000	3.0771	−0.1771	0.5370
				7	3.5000	3.6561	−0.1561	0.5790
				8	4.4000	4.2279	0.1721	0.5718
				9	4.7000	4.7261	−0.0261	0.4982
				10	5.1000	5.2945	−0.1945	0.5683
				11	5.7000	5.8629	−0.1629	0.5685
				12	6.6000	6.4024	0.1976	0.5395
				13	6.9000	6.9431	−0.0431	0.5407
				14	7.3000	7.4992	−0.1992	0.5561
				15	7.9000	8.0574	−0.1574	0.5582
				16	8.8000	8.6000	0.2000	0.5426
0.4	0.6	0.9	0.3	0	0.0000	0.0167	−0.0167	NA
				1	0.4000	0.4694	−0.0694	0.4527
				2	1.0000	1.1840	−0.1840	0.7147
				3	1.9000	1.7450	0.1550	0.5610
				4	2.2000	2.2281	−0.0281	0.4831
				5	2.6000	2.7778	−0.1778	0.5497
				6	3.2000	3.3501	−0.1501	0.5724
				7	4.1000	3.9405	0.1595	0.5903
				8	4.4000	4.4278	−0.0278	0.4873
				9	4.8000	4.9864	−0.1864	0.5586
				10	5.4000	5.5677	−0.1677	0.5814
				11	6.3000	6.1070	0.1930	0.5393
				12	6.6000	6.6364	−0.0364	0.5294
				13	7.0000	7.2009	−0.2009	0.5645
				14	7.6000	7.7564	−0.1564	0.5555
				15	8.5000	8.2974	0.2026	0.5410
				16	8.8000	8.8416	−0.0416	0.5442

(6.6) with E_n given exactly by (6.3), one can easily understand the quasi-periodic behaviour displayed by $E_n - \tilde{E}_n$ for every p levels in the large-n regime.

Meanwhile, the observation that the accuracies of the energy levels obtained from the SWKB approximation do not improve with increasing n can also be explained as follows. Similar to the traditional WKB approximation, the SWKB approximation is constructed for smooth potentials. However, for potentials of CSIPs with period greater than 2, rapid oscillations with increasing shorter periods play a dominant role at large x. As a consequence, the potentials cannot be considered as smooth even in the high-energy limit. Thus, the SWKB approximation is no longer a good approximation for such situations.

It is also worthy of remark that BSUSY also occurs for period-4 (or other even-period) CSIPs. In figure 4, the parameters $(\omega_1, \omega_2, \omega_3, \omega_4)$ are given by $(0.3, 0.4, 0.6, 0.9)$ for the solid line, and $(0.4, 0.6, 0.9, 0.3)$ for the dotted line. These two cases are a pair of supersymmetric partner potentials. While the values of the superpotential evaluated at the two

Table 4. A table showing the exact energy E_n and the SWKB energy $\tilde{E}_n$ for period-5 CSIPs with different values of $\omega_1, \omega_2, \omega_3, \omega_4$ and ω_5.

ω_1	ω_2	ω_3	ω_4	ω_5	n	E_n	$\tilde{E}_n$	$E_n - \tilde{E}_n$	$\tilde{E}_n - \tilde{E}_{n-1}$
0.4	0.5	0.55	1.0	1.1	0	0.0000	0.0000	0.0000	NA
					1	0.4000	0.3990	0.0010	0.3990
					2	0.9000	0.9105	−0.0105	0.5115
					3	1.4500	1.7194	−0.2694	0.8089
					4	2.4500	2.4622	−0.0122	0.7427
					5	3.5500	3.1779	0.3721	0.7157
					6	3.9500	3.9133	0.0367	0.7354
					7	4.4500	4.6127	−0.1627	0.6994
					8	5.0000	5.2725	−0.2725	0.6598
					9	6.0000	6.0079	−0.0079	0.7354
					10	7.1000	6.7174	0.3826	0.7095
					11	7.5000	7.4247	0.0753	0.7073
					12	8.0000	8.1487	−0.1487	0.7240
					13	8.5500	8.8606	−0.3106	0.7119
					14	9.5500	9.5538	−0.0038	0.6932
					15	10.6500	10.2508	0.3992	0.6970
					16	11.0500	10.9726	0.0774	0.7218
					17	11.5500	11.6949	−0.1449	0.7223
					18	12.1000	12.4085	−0.3085	0.7136
					19	13.1000	13.1055	−0.0055	0.6970
					20	14.2000	13.8195	0.3805	0.7141
0.4	0.5	1.0	0.6	0.7	0	0.0000	0.0000	0.0000	NA
					1	0.4000	0.4187	−0.0187	0.4187
					2	0.9000	1.0432	−0.1432	0.6245
					3	1.9000	1.8218	0.0782	0.7786
					4	2.5000	2.4306	0.0694	0.6088
					5	3.2000	3.0865	0.1135	0.6560
					6	3.6000	3.6948	−0.0948	0.6082
					7	4.1000	4.3254	−0.2254	0.6306
					8	5.1000	4.9608	0.1392	0.6354
					9	5.7000	5.6269	0.0731	0.6662
					10	6.4000	6.2731	0.1269	0.6462
					11	6.8000	6.9099	−0.1099	0.6367
					12	7.3000	7.5196	−0.2196	0.6097
					13	8.3000	8.1681	0.1319	0.6485
					14	8.9000	8.8312	0.0688	0.6631
					15	9.6000	9.4659	0.1341	0.6348
					16	10.0000	10.0902	−0.0902	0.6242
					17	10.5000	10.7295	−0.2295	0.6393
					18	11.5000	11.3838	0.1162	0.6543
					19	12.1000	12.0201	0.0799	0.6363
					20	12.8000	12.6574	0.1426	0.6373

spatial boundaries $x = 0$ and $x = \infty$ are of different signs for the former, which is a unbroken SUSY case; they are both positive for the latter, signifying BSUSY. Correspondingly, as shown in table 3, $\tilde{E}_0$ is exact (inexact) for the former (latter). Besides, both the superpotential and the potential of even-period CSIPs are singular at the origin. Numerical construction of such potentials is a bit tricky and, for the purpose of reference, we have included an appendix to the present paper to summarize relevant procedures.

7. Discussion and conclusion

In this paper, we have discussed the accuracy of the SWKB approximation for various CSIPs. The most interesting situation is the case of period-2 CSIPs, where the SWKB approximation can lead to exact eigenenergies of all even or odd states. Such alternate exactness of the SWKB approximation is attributable to two special features of period-2 CSIPs. (i) Period-2 CSIPs are equivalent to three-dimensional harmonic potentials, which are TSIPs; and (ii) the half-line problem associated with period-2 CSIPs, which are symmetric (or antisymmetric) about the origin, can give rise to either unbroken or broken SUSY. As the SWKB quantization rules (1.11) and (5.1), valid respectively for unbroken or broken SUSY, are known to yield exact results for TSIPs [20], the alternate exactness of the SWKB approximation for period-2 CSIPs can be explained.

The SWKB quantization rule can no longer lead to exact results for CSIPs with period p greater than 2. Actually, the difference between the exact and the SWKB results, in general, does not decrease significantly in the high-energy limit. Instead, it reveals a quasi-periodic variation of period p. Such findings are shown to be the direct consequence of the asymptotic behaviour of the potential at large x, which resembles that of a harmonic potential plus rapid oscillations with increasingly short periods (see figures 3–5).

Lastly, we also note that BSUSY can occur in even-period CSIPs. In such cases, the SWKB quantization rule (1.11) fails to yield exact ground-state energy, which is zero by convention in SQM. However, similar to the case of period-2 CSIPs, such ground states can be deleted due to the singular nature of associated wavefunctions at the origin.

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Appendix

A note is made here to discuss the construction of the superpotential of period-4 CSIPs and other even-periods CSIPs as well. As remarked above, superpotentials of CSIP with period $p > 3$ cannot be obtained analytically. For odd-period CSIPs, systems of ordinary differential equations like equation (6.1) can easily be written and hence the superpotentials can be obtained by numerical method with the nodal boundary condition $W = 0$ at $x = 0$ [25]. However, the situation becomes more difficult for CSIPs with an even period. In the following, we consider a period-4 CSIP as an example to illustrate the problems encountered in solving for the superpotentials numerically.

The shape invariant condition for period-4 CSIPs yields a set of differential equations for the superpotentails [25],

$$\begin{aligned} W_1^2 + W_1' &= W_2^2 - W_2' + \omega_1, \\ W_2^2 + W_2' &= W_3^2 - W_3' + \omega_2, \\ W_3^2 + W_3' &= W_4^2 - W_4' + \omega_3, \\ W_4^2 + W_4' &= W_1^2 - W_1' + \omega_4. \end{aligned} \tag{A.1}$$

When solving for the unknowns W_1', W_2', W_3' and W_4' from these four equations, it can be easily shown that the determinant of such a linear system vanishes identically, resulting in an undetermined case. Such behaviour prevails for any CSIPs with even periods greater than 2.

To overcome this problem, we first obtain a constraint equation from (A.1) as suggested in [25],

$$W_1^2 - W_2^2 + W_3^2 - W_4^2 = \frac{(\omega_1 - \omega_2 + \omega_3 - \omega_4)}{2}. \tag{A.2}$$

From (A.2), and its derivative,

$$W_1 W_1' - W_2 W_2' + W_3 W_3' - W_4 W_4' = 0, \tag{A.3}$$

both W_4 and W_4' can be expressed as functions of W_1, W_2 and W_3. Putting W_4 and W_4' into the first three equations of (A.1), we can rewrite them in a compact matrix form $\mathbf{MW}' = \mathbf{C}$, with

$$\mathbf{W}' = \begin{pmatrix} W_1' \\ W_2' \\ W_3' \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} W_2^2 - W_1^2 + \omega_1 \\ W_3^2 - W_2^2 + \omega_2 \\ W_1^2 - W_2^2 - (\omega_1 - \omega_2 - \omega_3 - \omega_4)/2 \end{pmatrix} \tag{A.4}$$

and

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \frac{W_1}{W_4} & -\frac{W_2}{W_4} & \frac{W_3}{W_4} + 1 \end{pmatrix}. \tag{A.5}$$

Therefore, a system of linear ordinary differential equations for W_1, W_2 and W_3 are obtained explicitly,

$$\mathbf{W}' = \mathbf{M}^{-1}\mathbf{C}, \tag{A.6}$$

which can be solved with standard numerical methods provided that appropriate boundary condition is given. Unlike the case of odd-period CSIPs, for CSIPs with even periods the superpotential becomes singular at origin $x = 0$. In fact, as shown in [25], for period-4 CSIPs the superpotential assume the following form near the origin:

$$W_1 = \frac{(\omega_1 - \omega_2 + \omega_3 - \omega_4)}{2\Omega_4}\frac{1}{x} + \frac{\Omega_4(\omega_1\omega_4 + \omega_1\omega_2 + \omega_3\omega_4 - \omega_3\omega_2)}{4(\omega_1 + \omega_3)(\omega_2 + \omega_4)}x + \cdots, \tag{A.7}$$

where $\Omega_4 = \omega_1 + \omega_2 + \omega_3 + \omega_4$. The series expansion of W_2, W_3 and W_4 can be obtained from cyclic permutation of the ωs. Besides, similar expansions can also be obtained for higher even-period CSIPs.

Instead of using the origin as the starting point to integrate (A.6), a point x_0 near origin is chosen to initiate the numerical integration scheme. For a sufficiently small x_0, accurate values of the superpotentials there can be evaluated using the leading terms in (A.7). In tandem with this, equation (A.6) can be solved by numerical integration. Analogously, superpotentials of other even-period CSIPs can also be constructed.

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